

# On the refinements of Sturmian-measurable partitions

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## Abstract

In the dynamics of a rotation of the unit circle by an irrational angle  $\alpha \in (0, 1)$ , we study the evolution of partitions consisting of finite unions of left-closed right-open intervals whose endpoints belong to the past trajectory of the point 0. We show that the refinements of these partitions eventually coincide with the refinements of a preimage of the Sturmian partition, which consists of two intervals  $[0, 1 - \alpha)$  and  $[1 - \alpha, 1)$ . In particular, the refinements of the partitions eventually consist of connected sets, i.e., intervals. We reformulate this result in terms of Sturmian subshifts and injectivity of the sliding block codes defined on them.

*Keywords:*

Sturmian subshift, partition, irrational rotation, sliding block code

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## 1. Main results

The dynamics given by a mapping  $T$  from a set  $X$  to itself is often described using the coding of orbits of points with respect to some specific partition  $\mathcal{R}$  of  $X$ . This standard approach of symbolic dynamics involves an analysis of the evolution of the partition  $\mathcal{R}$  with respect to  $T$ , where by the evolution we mean the refining sequence of partitions  $\mathcal{R}^n$  for  $n \geq 1$ , defined as follows:

$$\mathcal{R}^n = \left\{ \bigcap_{k=0}^{n-1} T^{-k} R_k \mid R_0, \dots, R_{n-1} \in \mathcal{R} \right\} \setminus \{\emptyset\}.$$

The partition  $\mathcal{R}^n$  is the common refinement of  $\mathcal{R}$  and its preimages  $T^{-j} \mathcal{R}$  for all integers  $j$  such that  $1 \leq j < n$ . We call  $\mathcal{R}^n$  the  $n$ -th refinement of  $\mathcal{R}$ .

Focusing on the dynamics of an irrational rotation of the unit circle, the most studied partitions are those inducing Sturmian sequences. Given an irrational  $\alpha \in (0, 1)$ , the unit circle is the factor group  $X = \mathbb{R}/\mathbb{Z}$ , represented by the fundamental domain  $[0, 1)$ , and the rotation  $T$  is the transformation of  $X$  given by the formula  $T(x) = (x + \alpha) \bmod 1$ . The partition inducing Sturmian sequences consists of two intervals  $P_0 = [0, 1 - \alpha)$  and  $P_1 = [1 - \alpha, 1)$ . We call this partition *Sturmian* and denote it by  $\mathcal{P}$ , i.e.,  $\mathcal{P} = \{P_0, P_1\}$ .

The evolution of the partition  $\mathcal{P}$  is closely related to combinatorial and other properties of the Sturmian sequence obtained as a coding of the orbit of the point 0 with respect to  $\mathcal{P}$  (for detailed study of Sturmian sequences see [Fog02], [Kũ03] or [MH40]). It is well known that the refinement  $\mathcal{P}^n$ , for  $n \in \mathbb{N}$ , consists of  $n + 1$  intervals in which the points  $T^{-k}(0)$ , for  $0 \leq k \leq n$ , divide the unit circle. The Three lengths theorem, due to Sós ([Sós58]), claims that these intervals are of two or three lengths. The theorem also describes these lengths in terms of convergents of  $\alpha$ . Because of the trivial identity  $(\mathcal{P}^m)^n = \mathcal{P}^{m+n-1}$ , the evolution of any refinement  $\mathcal{P}^m$  is covered by the mentioned results as well.

There is much less known about the evolution of other partitions. Combinatorial results for coding with respect to two-interval or finite-interval partitions with arbitrary endpoints were obtained in [Did98], [Ale96] and [AB98].

In this paper, we would like to introduce another class of partitions whose evolution can be surprisingly well described. The class consists of all partitions whose elements are finite unions of right-closed left-open intervals with endpoints from the set of preimages of the zero  $T^{-i}(0)$  for  $i \in \mathbb{N}$ . Partitions from this class are closely related to the partition  $\mathcal{P}$ , namely  $\mathcal{R}$  belongs to the class if and only if  $\mathcal{R}$  is rougher than  $\mathcal{P}^n$  for some  $n \in \mathbb{N}$ . In other words, a partition from the class consists of the sets that belongs to the algebra of sets generated by  $\mathcal{P}^n$ , for some  $n$  (the sets are  $\mathcal{P}^n$ -measurable). This is the reason to call these partitions *Sturmian-measurable* in the title of the article and throughout the paper.

Since the partition  $\mathcal{P}$  and its preimages  $T^{-j}\mathcal{P}$ ,  $j \in \mathbb{N}$ , generate the  $\sigma$ -field of Borel subsets of the unit interval, the class of all Sturmian-measurable partitions has the following interesting property; it is a dense set among all Borel partitions with respect to Rokhlin distance or entropy distance. Our main result shows that the refinements of any partition from the class coincide with the refinements of some preimage of  $\mathcal{P}$ .

In the following theorem, we introduce the main result of our paper. It concerns the refinements of Sturmian-measurable partitions that are non-trivial, i.e., consisting of at least two sets.

**Theorem 1.** *Let  $n \in \mathbb{N}$ . If  $\mathcal{R}$  is a non-trivial partition rougher than  $\mathcal{P}^n$ , then*

$$\mathcal{R}^k = T^{-\ell} \mathcal{P}^m = (T^{-\ell} \mathcal{P})^m, \quad \text{for some } k, \ell, m \in \mathbb{N} \text{ such that } \ell < n.$$

*In other words,  $\mathcal{R}^k$  is the partition of the unit circle into a union of right-closed left-open intervals whose endpoints are the preimages of zero  $T^{-i}(0)$  for  $i \in \mathbb{N}$  such that  $\ell \leq i \leq \ell + m$ .*

Let us notice that whenever the partition  $\mathcal{R}^k$  equals  $(T^{-\ell} \mathcal{P})^m$ , then for  $i \in \mathbb{N}$  every higher refinement  $\mathcal{R}^{k+i}$  equals the higher refinement  $(T^{-\ell} \mathcal{P})^{m+i}$ . In this case, the sequences  $(\mathcal{R}^k)_{k \in \mathbb{N}}$  and  $((T^{-\ell} \mathcal{P})^m)_{m \in \mathbb{N}}$  have the same tail.

The least  $k$  such that the partition  $\mathcal{R}^k$  is of the form described in Theorem 1 strongly depends on  $\mathcal{R}$ . In the next theorem, we provide an upper bound for the power  $k$  in terms of convergents of  $\alpha$ . The continued fraction expansion of  $\alpha$  is the following:

$$\alpha = [c_1, c_2, c_3, \dots] = \frac{1}{c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \dots}}}, \quad c_i \in \mathbb{N} \setminus \{0\}.$$

The convergents of  $\alpha$  are then  $\frac{p_k}{q_k}$  where  $p_k = c_k p_{k-1} + p_{k-2}$ ,  $p_0 = 0, p_1 = 1$  and  $q_k = c_k q_{k-1} + q_{k-2}$ ,  $q_0 = 1, q_1 = c_1$ . Denote  $r_k = q_k + q_{k-1}$  for  $k \geq 1$  and  $r_0 = 1$ .

**Theorem 2.** *Let  $\mathcal{R}$  be a Sturmian-measurable non-trivial partition. Let  $\ell$  be the largest positive integer and  $n$  the least positive integer such that  $\mathcal{R}$  is rougher than  $T^{-\ell} \mathcal{P}^n$ . If  $k \in \mathbb{N}$  such that  $r_{k-1} \leq n < r_k$ , then*

$$\mathcal{R}^{r_{k+3}+2r_k-n-2} = T^{-\ell} \mathcal{P}^{r_{k+3}+2r_k-3} = (T^{-\ell} \mathcal{P})^{r_{k+3}+2r_k-3}.$$

*Remark 3.* The numbers  $\ell$  and  $n$  in the statement of the last theorem are unambiguous. We will see this fact at the end of Section 2.2.

Let us notice that the theorem does not give the optimal answer when we look for the minimal  $m$  such that  $\mathcal{R}^m$  contains no disconnected set. It is neither optimal when searching for the minimal  $m$  such that  $\mathcal{R}^m$  is of the form  $T^{-i} \mathcal{P}^j$  for  $i, j \in \mathbb{N}$ . Indeed, when  $\mathcal{R}$  equals  $\mathcal{P}^n$  for some  $n$ , the optimal answer for both problems is  $m = 1$ , whereas the theorem suggests a number greater than  $r_{k+3} + r_k - 1$ .

In Section 4, we rephrase the results in terms of symbolic dynamics (see Propositions 12, 13 and 17). Sections 2 and 3 are devoted to the proofs of Theorem 1 and 2. In Section 5, we discuss possible generalizations of our results.

## 2. Preliminaries

Let  $X$  be a set and  $T : X \rightarrow X$  be a mapping on it. A *partition*  $\mathcal{R}$  of the space  $X$  is a set of non-empty pairwise disjoint sets from  $X$  such that they cover the whole set  $X$ , i.e.,  $X = \bigcup_{R \in \mathcal{R}} R$ . We say that a partition  $\mathcal{R}$  is *finer than* a partition  $\mathcal{R}'$  (or equivalently we say that  $\mathcal{R}'$  is *rougher than*  $\mathcal{R}$ ) if every  $R \in \mathcal{R}$  is a subset of a set  $R' \in \mathcal{R}'$ . In other words, every set from  $\mathcal{R}'$  is a union of sets from  $\mathcal{R}$ . This relation, denoted by  $\mathcal{R} > \mathcal{R}'$ , forms a lattice structure on the set of all partitions of  $X$ . The *supremum* of two partitions is denoted by  $\vee$ , it is also called the *join*, and for two partitions  $\mathcal{R}$  and  $\mathcal{R}'$  is defined as follows:

$$\mathcal{R} \vee \mathcal{R}' = \{R \cap R' \mid R \in \mathcal{R}, R' \in \mathcal{R}'\} \setminus \{\emptyset\}.$$

It is readily seen that  $\mathcal{R}^n = \bigvee_{i=0}^{n-1} T^{-i} \mathcal{R}$  for every partition  $\mathcal{R}$  and every  $n \geq 1$ .

Let a partition  $\mathcal{R}$  be labeled by indices forming a set  $\Sigma$ , i.e.,  $\mathcal{R} = \{R_i \mid i \in \Sigma\}$ . The *labeling* of the partition  $\mathcal{R}$  can be described by the mapping  $\phi_{\mathcal{R}} : X \rightarrow \Sigma$ , where  $\phi_{\mathcal{R}}(x) = i$  if  $x \in R_i$ . The sequence  $(\phi_{\mathcal{R}}(T^i x))_{i=0}^n$  is called the  $\mathcal{R}$ -*name* of  $x$  of length  $n$ . We get

$$\mathcal{R}^n = \{R_u \mid u \in \Sigma^n\} \setminus \{\emptyset\}, \quad \text{where } R_u = \bigcap_{k=0}^{n-1} T^{-k} R_{u_k}.$$

In other words, the partition  $\mathcal{R}^n$  is the partition induced by the  $\mathcal{R}$ -names of the points from  $X$  of length  $n$ , i.e., two points from  $X$  are in the same set from  $\mathcal{R}^n$  if and only if they have the same  $\mathcal{R}$ -name of length  $n$ .

For a given non-empty set  $A \subset X$  we denote the restriction to  $A$  of a partition  $\mathcal{R}$  by  $\mathcal{R}|A$ , i.e.,  $\mathcal{R}|A = \{R \cap A \mid R \in \mathcal{R}\} \setminus \{\emptyset\}$ . For  $i \leq j$ , denote the following family of sets:

$$\Lambda(A, i, j) = \{T^{-m}A \mid i \leq m < j\}.$$

If the sets in  $\Lambda(A, i, j)$  are pairwise disjoint, we call  $\Lambda(A, i, j)$  a *Rokhlin tower* (or simply a tower). The set  $T^{-i}A$  is called the *base of the Rokhlin tower*,  $T^{-(j-1)}A$  is called the *top* and  $j-i$  is the *height* of the Rokhlin tower. The set  $T^{-k}A$ ,  $i \leq k < j$ , is referred to as  $(k-i)$ -th *level* of the tower. We say that a word  $u = u_0u_1 \dots u_{j-i-1} \in \Sigma^{j-i}$  is the  *$\mathcal{R}$ -code of the tower* if

$$T^{-(j-1)+k}A \subset R_{u_k}, \quad \text{for every } k \text{ such that } 0 \leq k < j-i.$$

Hence, the  $\mathcal{R}$ -code of the tower equals the  $\mathcal{R}$ -name of length  $j-i$  of any point from the top of the tower.

Now, we introduce some notation which helps us to deal with  $\mathcal{R}$ -names of the points. For  $n \in \mathbb{N}$ , a *word* (or *block*) of length  $n$  over a finite set  $\Sigma$  is any finite sequence  $u = u_0 \dots u_{n-1}$  of elements from  $\Sigma$ . The set of all words of length  $n$  is denoted by  $\Sigma^n$ , the length of  $u$  is denoted by  $|u|$ . The set of all words of all lengths is denoted by  $\Sigma^*$ , i.e.,  $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^n$ . For two words  $u, v \in \Sigma^*$  we define their *concatenation*, denoted simply by  $uv$ , as a word from  $\Sigma^{|u|+|v|}$  such that  $(uv)_i = u_i$  if  $i < |u|$ , and  $(uv)_i = v_{i-|u|}$  if  $|u| \leq i < |u| + |v|$ . The concatenation of  $k$  copies of a word  $u$  is denoted by  $u^k$ . For  $u \in \Sigma^*$ ,  $m, n \in \mathbb{N}$ ,  $m \leq n \leq |u|$ , we denote by  $u[m, n)$  the subword of  $u$  given by the interval  $[m, n)$ , i.e.,  $u[m, n) \in \Sigma^{n-m}$  and

$$u[m, n)_i = u_{m+i}, \quad \text{for every } i \text{ such that } 0 \leq i < n-m.$$

The *shift* mapping  $S : \Sigma^* \rightarrow \Sigma^*$  is defined by  $S(u) = u[1, |u|)$  for all  $u \in \Sigma^*$ .

The shift mapping extends to infinite sequences in the following way. Let  $\Sigma^{\mathbb{N}}$  be the product space of countably many copies of a finite discrete space  $\Sigma$ . The *shift* mapping  $S : \Sigma^{\mathbb{N}} \rightarrow \Sigma^{\mathbb{N}}$  is defined by the following equality:  $(S(x))_i = x_{i+1}$  for  $x = (x_i)_{i=0}^{+\infty} \in \Sigma^{\mathbb{N}}$  and  $i \in \mathbb{N}$ . The mapping is continuous on  $\Sigma^{\mathbb{N}}$  and the pair  $(\Sigma^{\mathbb{N}}, S)$  is a *full shift*. Given a closed  $S$ -invariant subset  $\Gamma \in \Sigma^{\mathbb{N}}$  (i.e.,  $S(\Gamma) \subset \Gamma$ ), the pair  $(\Gamma, S)$  is a topological dynamical system called *subshift*, where  $S$  is considered to be restricted to  $\Gamma$ .

A classical way to produce a subshift is the coding of an arbitrary mapping  $T : X \rightarrow X$  with respect to a finite partition of  $X$ . Let  $\mathcal{R} = \{R_i \mid i \in \Sigma\}$

denote a finite partition of  $X$  labeled by  $\Sigma$ . If  $\Phi_{\mathcal{R}} : X \rightarrow \Sigma^{\mathbb{N}}$  is a mapping which maps  $x$  to its *infinite  $\mathcal{R}$ -name*, i.e.,  $\Phi_{\mathcal{R}}(x) = (\phi_{\mathcal{R}}(T^i x))_{i=0}^{+\infty}$ , then the mapping  $\Phi_{\mathcal{R}}$  commutes with  $T$  and  $S$ :

$$\Phi_{\mathcal{R}} \circ T = S \circ \Phi_{\mathcal{R}}.$$

In particular, the set  $\Phi_{\mathcal{R}}(X)$  and its closure are both invariant under  $S$ . Hence  $(\overline{\Phi_{\mathcal{R}}(X)}, S)$  is a subshift.

### 2.1. Sturmian partition

From now on, the mapping  $T$  is as given in Introduction, i.e., it is the rotation of the unit circle by an irrational angle  $\alpha$ :  $T(x) = (x + \alpha) \bmod 1$  for all  $x \in \mathbb{R}/\mathbb{Z}$ . Also recall that  $\mathcal{P}$  is the partition consisting of two sets  $P_0$  and  $P_1$ , where  $P_0 = [0, 1 - \alpha)$  and  $P_1 = [1 - \alpha, 1)$ . Although both these objects depend on  $\alpha$ , we do not explicitly state this dependence to ease the notation.

As already stated, the mapping  $T$  and partition  $\mathcal{P}$  define a Sturmian subshift  $(\overline{\Phi_{\mathcal{P}}([0, 1))}, S)$ . In this section, we state some results concerning our interest: some specific refinements of the partition  $\mathcal{P}$ .

It is well-known that the partition  $\mathcal{P}^n$  consists of  $n + 1$  intervals. The Three lengths theorem (see [Sós58]) says that these intervals are of at most three lengths and specifies the lengths in terms of convergents. In particular, it is shown that the partition  $\mathcal{P}^{r_k-1}$ , for  $k \in \mathbb{N}$ , has intervals of just two lengths. In geometric proofs of the Three lengths theorem, not only the lengths of intervals are determined, but also their endpoints. A version of the theorem, which is needed later, is recalled as Proposition 4. It is a special case of the description of intervals from  $\mathcal{P}^n$  used in the proof of the Three lengths theorem in [Kü03] (Theorem 4.45, p. 160).

Before stating this version of the theorem, we need some more notations. First, let us notice that the numbers  $q_k \alpha - p_k$  for  $k \in \mathbb{N}$  form an alternating sequence and their absolute values  $\eta_k = |q_k \alpha - p_k|$  satisfy the implicit formula  $\eta_k = \eta_{k-2} - c_k \eta_{k-1}$ ,  $\eta_0 = \alpha$ , and  $\eta_1 = 1 - c_1 \alpha$ . Denote the following sequence of intervals in  $\mathbb{R}/\mathbb{Z}$  as follows:

$$I_k = \begin{cases} [0, \eta_k) & \text{for } k \text{ even,} \\ [-\eta_k, 0) & \text{for } k \text{ odd.} \end{cases}$$

**Proposition 4.** *Let  $k \in \mathbb{N}$ . The partition  $\mathcal{P}^{r_k-1}$  consists of two Rokhlin towers  $\Lambda(I_k, 0, q_{k-1})$  and  $\Lambda(I_{k-1}, 0, q_k)$ , i.e.*

$$\mathcal{P}^{r_k-1} = \Lambda(I_k, 0, q_{k-1}) \cup \Lambda(I_{k-1}, 0, q_k), \quad \text{for } k \geq 1.$$

Moreover,

$$T^{-(q_{k-1}+sq_k)}I_k \subset I_{k-1}, \quad \text{for every } s \text{ such that } 0 \leq s < c_{k+1}.$$

We also sometimes say that the towers  $\Lambda(I_k, 0, q_{k-1})$  and  $\Lambda(I_{k-1}, 0, q_k)$  form the partition  $\mathcal{P}^{r_k-1}$ .

We conclude this section with an iterative formula for  $\mathcal{R}$ -codes of the towers from the previous proposition.

**Lemma 5.** *Let  $k \in \mathbb{N}$  and  $\mathcal{R}$  be a partition indexed by  $\Sigma$ . If  $u \in \Sigma^{q_{k-1}}$  and  $v \in \Sigma^{q_k}$  are the  $\mathcal{R}$ -codes of the towers  $\Lambda(I_k, 0, q_{k-1})$  and  $\Lambda(I_{k-1}, 0, q_k)$ , respectively, then  $v$  and  $v^{c_{k+1}}u$  are the  $\mathcal{R}$ -codes of the towers  $\Lambda(I_{k+1}, 0, q_k)$  and  $\Lambda(I_k, 0, q_{k+1})$ , respectively.*

The proof of the lemma is a straightforward application of Proposition 4.

The previous lemma and proposition are illustrated in Figures 1, 2, 3 and 4. The first three figures depict the two towers  $\Lambda(I_k, 0, q_{k-1})$  and  $\Lambda(I_{k-1}, 0, q_k)$ , the first being always on the left. Figure 1 shows the partition  $\mathcal{P}^{r_k-1}$  as two towers. Therein and in what follows, we use a compact notation  $\langle n \rangle := T^{-n}(0)$ . In Figure 2, the  $\mathcal{R}$ -codes of the towers are graphically presented. A level is labeled by the symbol  $a \in \Sigma$  if and only if it is included in  $R_a$ . The word  $u = u_0 \cdots u_{q_{k-1}-1}$  is the  $\mathcal{R}$ -code of  $\Lambda(I_k, 0, q_{k-1})$  and  $v = v_0 \cdots v_{q_k-1}$  of  $\Lambda(I_{k-1}, 0, q_k)$ . Figure 3 shows the dynamics given by  $T$ . The arrows indicate that each level, except the base, is mapped to the level below. In accordance with the dynamics, the  $\mathcal{R}$ -codes are written in the towers from the top to the base. The arrows also show where the points from the base are mapped by  $T$ . By Proposition 4 the right part of the base of the right tower of length  $\eta_k$  is mapped to the top of the left tower. The rest of the base and the base of the left tower must be mapped to the top of the right tower due to the injectivity of the mapping  $T$ . Figure 4 illustrates two consecutive iterations of  $\mathcal{R}$ -codes given by the last lemma.

## 2.2. Sturmian-measurable partitions

Let us recall that a partition of  $[0, 1)$  is Sturmian-measurable if it is a finite partition whose elements are finite unions of right-closed left-open intervals with endpoints from the set of preimages of zero  $T^{-i}(0)$ ,  $i \in \mathbb{N}$ . The class of all Sturmian-measurable partitions is closed under taking preimages and joins. In particular, for all  $m \in \mathbb{N}$  the partition  $\mathcal{R}^m$  is Sturmian-measurable whenever  $\mathcal{R}$  is Sturmian-measurable.

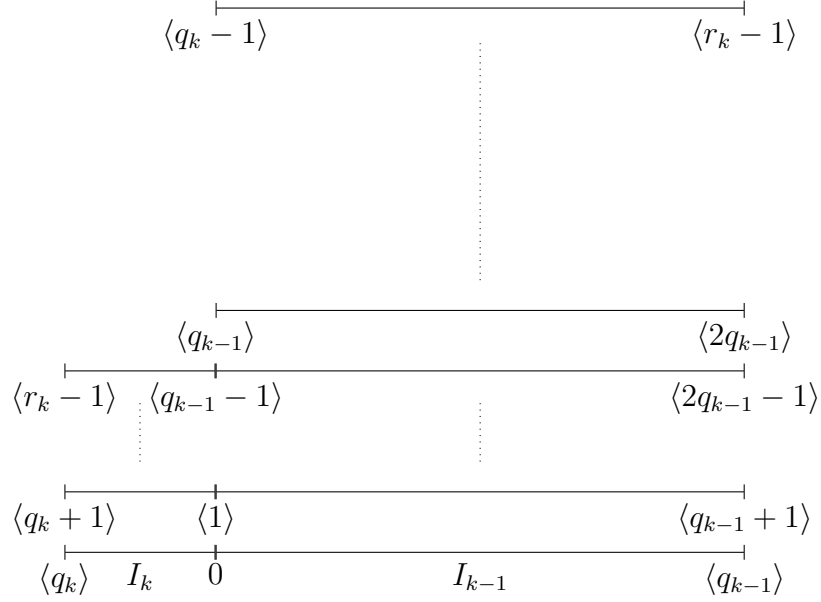


Figure 1: Rokhlin towers  $\Lambda(I_k, 0, q_{k-1})$  and  $\Lambda(I_{k-1}, 0, q_k)$  forming the partition  $\mathcal{P}^{r_k-1}$ .

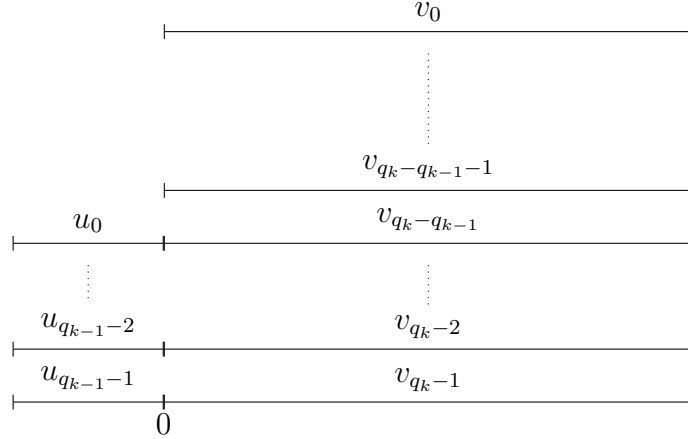


Figure 2:  $\mathcal{R}$ -codes of the towers forming  $\mathcal{P}^{r_k-1}$ , word  $u = u_0 \cdots u_{q_{k-1}-1}$  is the  $\mathcal{R}$ -code of  $\Lambda(I_k, 0, q_{k-1})$ , word  $v = v_0 \cdots v_{q_k-1}$  is the  $\mathcal{R}$ -code of  $\Lambda(I_{k-1}, 0, q_k)$ .

For a partition  $\mathcal{R}$  we define the set  $\partial \mathcal{R}$  as the union of the boundaries of the sets from  $\mathcal{R}$ . In this definition we consider the topology of  $\mathbb{R}/\mathbb{Z}$  represented by the fundamental domain  $[0, 1)$ . The elements of  $\partial \mathcal{R}$  are *cutpoints*



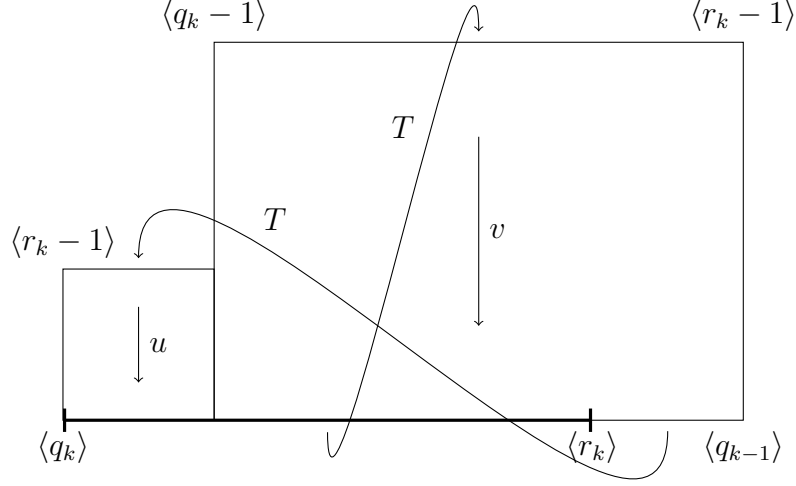
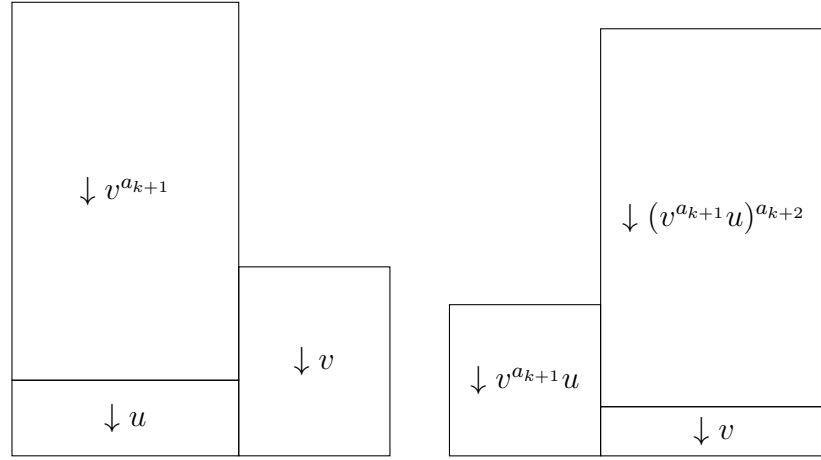


Figure 3: Dynamics (given by  $T$ ) on the towers  $\Lambda(I_k, 0, q_{k-1})$  and  $\Lambda(I_{k-1}, 0, q_k)$ .



(a)  $\mathcal{R}$ -codes of the towers  $\Lambda(I_k, 0, q_{k+1})$  and  $\Lambda(I_{k+1}, 0, q_k)$  forming  $\mathcal{P}^{r_{k+1}-1}$ .  
(b)  $\mathcal{R}$ -codes of the two towers  $\Lambda(I_{k+2}, 0, q_{k+1})$  and  $\Lambda(I_{k+1}, 0, q_{k+2})$  forming  $\mathcal{P}^{r_{k+2}-1}$ .

Figure 4

of the partition  $\mathcal{R}$ . Put

$$\mathcal{C}(\mathcal{R}) = \{i \in \mathbb{Z} \mid T^{-i}(0) \in \partial \mathcal{R}\}.$$

The numbers from this set are called *cut-indices* of  $\mathcal{R}$ . The terminology follows the fact that for Sturmian-measurable partition  $\mathcal{R}$ , the set  $\partial \mathcal{R}$  is the smallest set such that the sets in  $\mathcal{R}$  can be described as a finite union of intervals whose endpoints belong to  $\partial \mathcal{R}$ . Hence, the partition  $\mathcal{R}$  cuts the circle just at the points from  $\partial \mathcal{R}$ .

The cutpoints and cut-indices interplay with the lattice operations and the transformation  $T$  in the following manner:

$$\begin{aligned}\partial(\mathcal{R} \vee \mathcal{R}') &= \partial \mathcal{R} \cup \partial \mathcal{R}', & \partial(T^{-j} \mathcal{R}) &= T^{-j}(\partial \mathcal{R}), \\ \mathcal{C}(\mathcal{R} \vee \mathcal{R}') &= \mathcal{C}(\mathcal{R}) \cup \mathcal{C}(\mathcal{R}'), & \mathcal{C}(T^{-j} \mathcal{R}) &= j + \mathcal{C}(\mathcal{R}),\end{aligned}$$

for every integer  $j$  and partitions  $\mathcal{R}$  and  $\mathcal{R}'$ . In particular,

$$\begin{aligned}\min(\mathcal{C}(T^{-i}(\mathcal{R}^j))) &= i + \min(\mathcal{C}(\mathcal{R})), \\ \max(\mathcal{C}(T^{-i}(\mathcal{R}^j))) &= i + j - 1 + \max(\mathcal{C}(\mathcal{R})),\end{aligned}$$

for every  $i \in \mathbb{Z}$  and  $j \in \mathbb{N}$ .

Let us reformulate the assumption of Theorem 2 using cut-indices. Given a Sturmian-measurable partition  $\mathcal{R}$ , integers  $s, m \in \mathbb{N}$ , the partition  $T^{-s} \mathcal{P}^m$  is finer than  $\mathcal{R}$  if and only if the cut-indices of  $\mathcal{R}$  belong to the interval  $[s, s + m]$ . If  $\mathcal{R}$  is non-trivial, the smallest interval containing all the cut-indices is the interval  $[\min(\mathcal{C}(\mathcal{R})), \max(\mathcal{C}(\mathcal{R}))]$ . Hence, among all pairs  $(s, m)$  such that  $T^{-s} \mathcal{P}^m$  is finer than  $\mathcal{R}$ , there is a pair

$$(s', m') = (\min(\mathcal{C}(\mathcal{R})), \max(\mathcal{C}(\mathcal{R})) - \min(\mathcal{C}(\mathcal{R}))),$$

which maximizes the first coordinate and minimizes the second simultaneously. Hence the largest  $\ell$  and the least  $n$  introduced in Theorem 2 are uniquely determined by the partition  $\mathcal{R}$ . The number  $\ell$  equals the minimal cut-index of  $\mathcal{R}$  and  $n$  equals the difference between the maximal and the minimal cut-index of  $\mathcal{R}$ .

### 3. Proof of the main result

In this section we prove Theorem 2 by applying Proposition 6 which is stated below. In the second part we prove the proposition itself using the analysis of a periodic structure in  $\mathcal{R}$ -codes. Let us notice, that the proposition is a special case of Theorem 2.

**Proposition 6.** *Let  $\mathcal{R}$  be rougher than  $\mathcal{P}^{r_k-1}$  for some  $k \in \mathbb{N}$ . If 0 and  $r_k - 1$  are cut-indices of  $\mathcal{R}$ , then*

$$\mathcal{R}^{r_{k+3}+r_k-1} = \mathcal{P}^{r_{k+3}+2r_k-3}.$$

*Proof of Theorem 2.* Let  $\ell, n \in \mathbb{N}$  and a partition  $\mathcal{R}$  satisfy the assumptions of Theorem 2, i.e.,  $\ell = \min(\mathcal{C}(\mathcal{R}))$ ,  $n = \max(\mathcal{C}(\mathcal{R})) - \min(\mathcal{C}(\mathcal{R}))$ , and  $k \in \mathbb{N}$  such that  $r_{k-1} \leq n < r_k$ . Denote  $\mathcal{R}' = T^\ell(\mathcal{R}^{r_k-n})$ . Thus, since  $\mathcal{R}$  is finer than  $T^\ell \mathcal{P}^m$ , we get

$$\mathcal{R}' < T^\ell \left( (T^{-\ell} \mathcal{P}^n)^{r_k-n} \right) = T^\ell (T^{-\ell} (\mathcal{P}^n)^{r_k-n}) = \mathcal{P}^{r_k-1}.$$

In particular,  $\mathcal{R}'$  is Sturmian-measurable.

By the properties of cut-indices, mentioned in the previous section,

$$\begin{aligned} \min(\mathcal{C}(\mathcal{R}')) &= -\ell + \min(\mathcal{C}(\mathcal{R})) = 0, \\ \max(\mathcal{C}(\mathcal{R}')) &= -\ell + r_k - n - 1 + \max(\mathcal{C}(\mathcal{R})) = -\ell + r_k - n - 1 + \ell + n = r_k - 1. \end{aligned}$$

Thus, the partition  $\mathcal{R}'$  satisfies the assumptions of Proposition 6. Applying the proposition we get

$$\begin{aligned} T^{-\ell} (\mathcal{P}^{r_{k+3}+2r_k-3}) &= T^{-\ell} ((\mathcal{R}')^{r_{k+3}+r_k-1}) = T^{-\ell} \left( (T^\ell \mathcal{R}^{r_k-n})^{r_{k+3}+r_k-1} \right) \\ &= \mathcal{R}^{r_{k+3}+2r_k-n-2}. \end{aligned}$$

To complete the proof we need to prove Proposition 6. □

### 3.1. Proof of Proposition 6

In the rest of this section we fix  $k \in \mathbb{N}$  and a partition  $\mathcal{R} = \{R_a, a \in \Sigma\}$  satisfying the assumptions of Proposition 6, i.e.,  $k \geq 1$ ,  $\mathcal{R}$  is rougher than  $\mathcal{P}^{r_k-1}$  and the indices 0 and  $r_k - 1$  are cut-indices of  $\mathcal{R}$ .

Let us denote by  $u$  and  $v$  the  $\mathcal{R}$ -codes of the towers  $\Lambda(I_k, 0, q_{k-1})$  and  $\Lambda(I_{k-1}, 0, q_k)$  respectively (see Figure 2). The condition on the cut-indices of  $\mathcal{R}$  can be rephrased into the following conditions on  $u$  and  $v$ :

- since the cutpoint 0 of  $\mathcal{R}$  is the common endpoint of the bases of the towers, the beginning of  $u$  and  $v$  differs, i.e.,  $u_0 \neq v_0$ ;
- since the cutpoint  $T^{-(r_k-1)}0$  is the common endpoint of the tops of the towers, the end of  $u$  and  $v$  differs, i.e.,  $u_{q_{k-1}-1} \neq v_{q_k-1}$ .

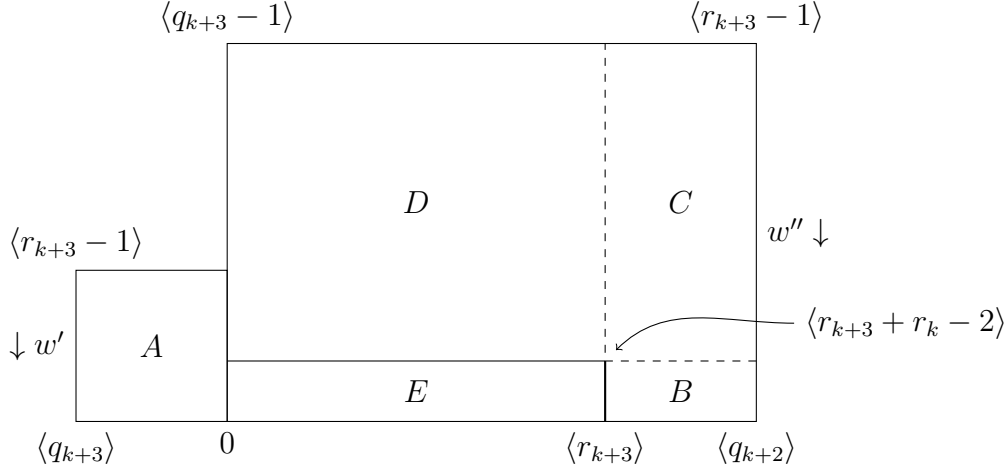


Figure 5: Important parts of  $\mathcal{P}^{r_{k+3}+r_k-2}$

Put

$$w = v^{c_{k+1}}u, \quad w' = (w)^{c_{k+2}}v, \quad w'' = (w')^{c_{k+3}}w, \quad z = w''w'.$$

Applying Lemma 5 three times we get that  $w$ ,  $w'$  and  $w''$  are the  $\mathcal{R}$ -codes of the towers  $\Lambda(I_k, 0, q_{k+1})$ ,  $\Lambda(I_{k+1}, 0, q_{k+2})$  and  $\Lambda(I_{k+2}, 0, q_{k+3})$ , respectively. It implies that the words  $w$  and  $w'$  are also the  $\mathcal{R}$ -codes of the towers  $\Lambda(I_{k+2}, 0, q_{k+1})$  and  $\Lambda(I_{k+3}, 0, q_{k+2})$ . We have just described the  $\mathcal{R}$ -codes of the towers which form the partitions  $\mathcal{P}^{r_{k+1}-1}$ ,  $\mathcal{P}^{r_{k+2}-1}$  and  $\mathcal{P}^{r_{k+3}-1}$ . Our aim is to use these  $\mathcal{R}$ -codes to describe  $\mathcal{R}$ -names of length  $r_{k+3}$  of the points from distinct sets from  $\mathcal{P}^{r_{k+3}+r_{k+1}-2}$ . The key role throughout this section will be played by the word  $z$ , which is the  $\mathcal{R}$ -code of the tower  $\Lambda(I_{k+3}, 0, r_{k+3})$ .

We divide the interval  $[0, 1)$  into several Rokhlin towers (see Figure 5) and separately analyze the  $\mathcal{R}$ -names of length  $r_{k+3}$  of the points from distinct towers.

Put

$$\begin{aligned} A &= \Lambda(I, 0, q_{k+2}), & B &= \Lambda(K, 0, r_k - 1), & C &= \Lambda(K, r_k - 1, q_{k+3}), \\ D &= \Lambda(J, r_k - 1, q_{k+3}), & E &= \Lambda(J, 0, r_k - 1), \end{aligned}$$

where  $I = I_{k+3}$ ,  $J = I_{k+2} \setminus T^{-q_{k+2}}I_{k+3}$  and  $K = T^{-q_{k+2}}I_{k+3}$ . The intervals  $I$ ,  $J$  and  $K$  are the bases of the towers  $A$ ,  $E$  and  $B$ , respectively. For any of the towers  $A, B, C, D$  and  $E$ , denote its union using the tilde over the letter, e.g.,  $\tilde{A} = \bigcup A$ .

According to the Three lengths theorem the partition  $\mathcal{P}^{r_{k+3}+r_k-2}$  arises from the partition  $\mathcal{P}^{r_{k+3}-1}$  by adding new cutpoints

$$\langle j \rangle, \quad \text{for } j \text{ such that } r_{k+3} \leq j \leq r_{k+3} + r_k - 2.$$

These points are illustrated by the vertical line between the towers  $B$  and  $E$  in Figure 5. It implies that

$$\mathcal{P}^{r_{k+3}+r_k-2} = A \cup B \cup E \cup \Lambda(I_{k+3}, r_k - 1, q_{k+3}).$$

For a point  $x \in [0, 1)$ , denote the  $\mathcal{R}$ -name of  $x$  of length  $r_{k+3}$  by  $\hat{x}$ . In addition, denote the addition and subtraction in the finite modular group  $\mathbb{Z}_{r_{k+3}}$  by  $\oplus$  and  $\ominus$ . For a word  $u \in \Sigma^{r_{k+3}}$ , put

$$Per(u) = \{j \in \mathbb{Z}_{r_{k+3}} \mid u_j = u_{j \ominus |w'|}\}, \quad \sigma(u) = u_{r_{k+3}-1} u_0 u_1 \dots u_{r_{k+3}-2} u_0 \in \Sigma^{r_{k+3}}.$$

Obviously,  $Per(\sigma(u)) = Per(u) \oplus 1$ .

To find  $Per(z)$  we need to compare  $z$  and  $\sigma^{|w'|}(z)$ ,

$$\begin{aligned} z &= w''w' = (w')^{c_{k+3}} ww' = (w')^{c_{k+3}} ww^{c_{k+2}}v = (w')^{c_{k+3}} w^{c_{k+2}}wv \\ &= (w')^{c_{k+3}} w^{c_{k+2}}v^{c_{k+1}}uv, \\ \sigma^{|w'|}(z) &= w'w'' = w'(w')^{c_{k+3}}w = (w')^{c_{k+3}}w'w = (w')^{c_{k+3}} w^{c_{k+2}}vw \\ &= (w')^{c_{k+3}} w^{c_{k+2}}vv^{c_{k+1}}u = (w')^{c_{k+3}} w^{c_{k+2}}v^{c_{k+1}}vu. \end{aligned}$$

One can see that the words coincide on first  $|z| - |v| - |u|$  positions. Since the beginnings and ends of  $u$  and  $v$  differ, we get that the words above differ at positions  $|z| - 1$  and  $|z| - |v| - |u|$ . Thus,

$$\mathbb{N} \cap [0, |z| - |v| - |u|) \subseteq Per(z)$$

and

$$\{|z| - 1, |z| - |v| - |u|\} \cap Per(z) = \emptyset.$$

**Lemma 7.** *If  $x \in \tilde{A} \cup \tilde{B} \cup \tilde{C}$ , i.e.,  $x \in T^{-m}I$  for some  $m \in [0, r_{k+3})$ , then*

$$\hat{x} = z[r_{k+3} - m - 1, r_{k+3})z[0, r_{k+3} - m - 1) = \sigma^{m+1-r_{k+3}}(z).$$

*In particular,*

$$Per(\hat{x}) = Per(z) \oplus (m \oplus 1 \ominus r_{k+3}).$$

*Proof.* First, let us prove that for  $x \in T^{-(q_{k+3}-1)}I_{k+2}$ ,  $\hat{x} = z$ . Since  $T^{-(q_{k+3}-1)}I_{k+2}$  is the top of the tower  $\Lambda(I_{k+2}, 0, q_{k+3})$ , we get that the beginning of  $\hat{x}$  equals the  $\mathcal{R}$ -code of the tower, i.e.,

$$\hat{x}[0, q_{k+3}) = w'' = z[0, q_{k+3}).$$

Put  $y = T^{q_{k+3}}x$ . Surely,  $\hat{x}[q_{k+3}, r_{k+3})$  equals the  $\mathcal{R}$ -name of  $y$  of length  $q_{k+2}$ . Since  $y \in T(I_{k+2})$ , where  $I_{k+3}$  is the base of the tower  $\Lambda(I_{k+2}, 0, q_{k+3})$ , the point  $y$  should be on the top of either of the towers  $\Lambda(I_{k+3}, 0, q_{k+2})$  and  $\Lambda(I_{k+2}, 0, q_{k+3})$ . In the former case, the  $\mathcal{R}$ -name of  $y$  of length  $q_{k+2}$  equals  $w'$ . In the latter case, the  $\mathcal{R}$ -name equals the beginning of  $w''$  of length  $q_{k+2}$ , i.e.,

$$\hat{x}[q_{k+3}, r_{k+3}) = w''[0, q_{k+2}) = w'.$$

Altogether,  $\hat{x} = w''w' = z$ .

We proceed with the proof of the lemma. Let  $x \in T^{-m}I$  for some  $0 \leq m < q_{k+3}$ . Put  $y' = T^{m-(q_{k+3}-1)}x$ ,  $y'' = T^{m+1}x$ . Thus,

$$y' \in T^{-(r_{k+3}-1)}I \subset T^{-(q_{k+3}-1)}I_{k+2}, \quad y'' \in T(I) \subset T^{-(q_{k+3}-1)}I_{k+2}.$$

(For the last inclusion see the discussion below Proposition 4). We get that  $\hat{y}'$  and  $\hat{y}''$  equals  $z$ . As follows immediately from the definition,

$$\begin{aligned} \hat{x}[0, m+1) &= \hat{y}'[r_{k+3}-m-1, r_{k+3}) = z[r_{k+3}-m-1, r_{k+3}), \\ \hat{x}[m+1, r_{k+3}) &= \hat{y}''[0, r_{k+3}-m-1) = [0, r_{k+3}-m-1), \end{aligned}$$

which concludes the proof.  $\square$

**Lemma 8.** *If  $x$  is from  $\tilde{E}$ , i.e.,  $x \in T^{-m}J$ ,  $0 \leq m < r_k - 1$ , then*

$$\hat{x} = w''[q_{k+3}-m-1, q_{k+3})w''w'[0, q_{k+2}-m-1)$$

*and neither  $m \oplus |w'|$  nor  $m \ominus r_k \oplus 1$  belong to  $Per(\hat{x})$ .*

*Proof.* Let  $x$  be from  $\tilde{E}$ , i.e.,  $x \in T^{-m}J$ ,  $0 \leq m < r_k - 1$ . Then  $T^{m-(r_{k+3}-1)}x$  belongs to the top of the tower  $\Lambda(I_{k+2}, 0, q_{k+3})$  whose  $\mathcal{R}$ -code is  $w''$ . Hence,  $\hat{x}[0, m+1)$  equals  $w''[q_{k+3}-m-1, q_{k+3})$ . Since the point  $y = T^{m+1}x$  belongs to the top of the tower  $\Lambda(I_{k+2}, 0, q_{k+3})$ ,  $\hat{y}$  equals  $z$  (see the proof of the previous proposition). It implies that

$$\begin{aligned} \hat{x}[m+1, r_{k+3}) &= \hat{y}[0, r_{k+3}-m-1) = z[0, r_{k+3}-m-1) \\ &= w''w'[0, q_{k+2}-m-1, q_{k+3}). \end{aligned}$$

We proved the first part of the lemma.

The equality above implies that  $\hat{x}_m$  is the last letter of  $w''$ . But  $w''$  ends with  $w$  and  $w$  ends with  $u$ . So,  $x_m$  is equal to the last letter of  $u$ . Moreover,

$$\hat{x}[m+1, m+|w'|+1) = w''[0, |w'|) = w' = w^{c_{k+2}}v.$$

Thus,  $\hat{x}_{m+|w'|}$  equals the last letter of  $v$  and so differs from  $x_m$ , i.e.,  $(m \oplus |w'|)$  does not belong to  $Per(\hat{x})$ .

Since  $m < r_k - 1$  and  $r_k < |w''| = q_{k+2}$ , we get

$$\hat{x}_{m \ominus r_k \oplus 1} = \hat{x}_{m+1+|w''|+|w'|-r_k} = w'_{|w'|-r_k}$$

and

$$\hat{x}_{m \ominus r_k \oplus 1 \ominus |w'|} = \hat{x}_{m+1+|w''|-r_k} = w''_{|w''|-r_k}.$$

However,

$$w' = w^{c_{k+2}}v = \overbrace{w^{c_{k+2}-1}v^{c_{k+1}}}^{|w'|-r_k}uv \quad \text{and} \quad w'' = (w')^{c_{k+3}}w = \overbrace{(w')^{c_{k+3}}v^{c_{k+1}-1}}^{|w''|-r_k}vu.$$

Since  $r_k = |u| + |v|$ ,  $\hat{x}_{m \ominus r_k \oplus 1}$  equals the first letter of  $u$  and  $\hat{x}_{m \ominus r_k \oplus 1 \ominus |w'|}$  equals the first letter of  $v$ . By the properties of  $u$  and  $v$ , the letters differ, i.e.,  $(m \ominus r_k \oplus 1)$  does not belong to  $Per(\hat{x})$ .  $\square$

**Lemma 9.**

- If  $x \in T^{-m}I$  and  $y \in T^{-m'}I$  for some  $0 \leq m < m' < r_{k+3}$ , then  $\hat{x} \neq \hat{y}$ .
- If  $x \in \tilde{A} \cup \tilde{B} \cup \tilde{C}$  and  $y \in \tilde{E}$ , then  $\hat{x} \neq \hat{y}$ .

*Proof.* Let  $x \in T^{-m}I$  and  $y \in T^{-m'}I$  for some  $0 \leq m < m' < r_{k+3}$ . Since  $r_{k+3} - r_k$  is greater than  $r_k/2$ , then

$$0 < m \ominus m' < r_{k+3} - r_k \quad \text{or} \quad 0 < m' \ominus m < r_{k+3} - r_k.$$

Suppose that the former inequality holds. Then the number

$$m \ominus r_{k+3} = m' \oplus (m \ominus m') \ominus r_{k+3}$$

does not belong to  $Per(\hat{x})$ , but belongs to  $Per(\hat{y})$  (see Lemma 7). It implies that  $\hat{x} \neq \hat{y}$ . By similar arguments, the latter of the above mentioned

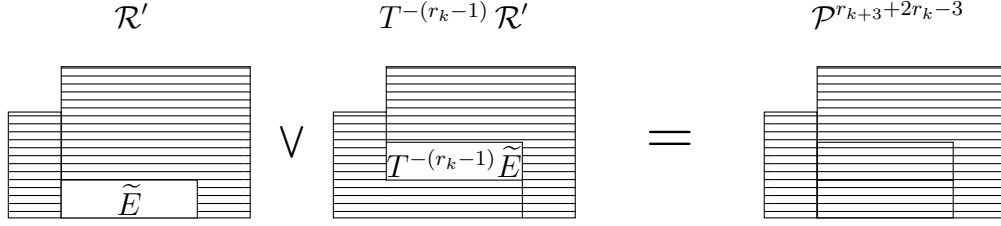


Figure 6: Partitions  $\mathcal{R}'$ ,  $T^{-(r_k-1)} \mathcal{R}'$  and  $\mathcal{P}^{r_{k+3}+2r_k-3}$ .

inequalities implies that  $m' \ominus r_{k+3}$  does not belong to  $Per(\hat{y})$ , but belongs to  $Per(\hat{x})$ . We get again that  $\hat{x}$  and  $\hat{y}$  differ.

Let the group  $\mathbb{Z}_{r_{k+3}}$  be equipped with the “circle” distance  $d(i, j)$  defined as the minimum of  $i \ominus j$  and  $j \ominus i$ . Let  $x \in \tilde{A} \cup \tilde{B} \cup \tilde{C}$  and  $y \in \tilde{E}$ , i.e.,  $y \in T^{-m}J$  for some  $m < r_k$ . By Lemma 7 we get that

$$\text{diam}(\mathbb{Z}_{r_{k+3}} \setminus Per(\hat{x})) \leq j \leq r_k,$$

and by Lemma 8 we deduce that

$$\text{diam}(\mathbb{Z}_{r_{k+3}} \setminus Per(\hat{y})) \geq d(m \oplus |w'|, m \ominus r_k \oplus 1) = \min(r_k \ominus 1 \oplus |w'|, 1 \ominus r_k \ominus |w'|).$$

Since  $r_k + |w'| \leq r_{k+3}$  and  $|w'| \geq 2$ , we get

$$\begin{aligned} r_k \ominus 1 \oplus |w'| &= r_k - 1 + |w'| > r_k \quad \text{and} \\ 1 \ominus r_k \ominus |w'| &= r_{k+3} + 1 - r_k - |w'| > q_{k+3} + q_{k+2} - q_k - q_{k-1} - q_{k+2} \\ &= q_{k+2} + q_{k+1} - q_k - q_{k-1} \geq q_{k+2} \geq r_{k+1} > r_k. \end{aligned}$$

Thus, the diameters of the above mentioned sets differ. It implies  $\hat{x} \neq \hat{y}$ .  $\square$

**Corollary 10.** *Partition  $\mathcal{R}^{r_{k+3}}$  is finer than  $\mathcal{R}'$ , where*

$$\mathcal{R}' = \{\tilde{E}\} \cup A \cup B \cup \Lambda(J \cup K, r_k - 1, q_{k+3}).$$

*Proof.* The previous lemma shows that

$$\mathcal{R}^{r_{k+3}}|_{\tilde{X} \setminus \tilde{D}} > \{\tilde{E}\} \cup A \cup B \cup C.$$

Since  $\mathcal{R}^{r_{k+3}} < \mathcal{P}^{r_{k+3}+r_k-2}$ , the points in  $\tilde{D}$  have the same  $\mathcal{R}$ -names of length  $r_{k+3}$  as the points from the same level in  $\tilde{C}$ . More precisely, if  $x \in T^{-j}J$ ,  $r_k - 1 \leq j < q_{k+3}$ , then  $\hat{x} = \hat{y}$  for any  $y \in T^{-m}I_{k+2}$ , in particular,  $\hat{x} = \hat{y}$  for any  $y \in T^{-j}K$ .  $\square$



**Proposition 11.**  $\mathcal{R}^{r_{k+3}+r_k-1} = \mathcal{P}^{r_{k+3}+2r_k-3}$ .

*Proof.* Since  $\mathcal{R} < \mathcal{P}^{r_k-1}$ ,

$$\mathcal{R}^{r_{k+3}+r_k-1} < (\mathcal{P}^{r_k-1})^{r_{k+3}+r_k} = \mathcal{P}^{r_{k+3}+2r_k-3}.$$

The opposite inequality arises as follows:

$$\begin{aligned} \mathcal{R}^{r_{k+3}+r_k-1} &= (\mathcal{R}^{r_{k+3}})^{r_k} > \mathcal{R}^{r_{k+3}} \vee T^{-(r_k-1)} \mathcal{R}^{r_{k+3}} \\ &> \mathcal{R}' \vee T^{-(r_k-1)} \mathcal{R}' > \mathcal{P}^{r_{k+3}+2r_k-3}. \end{aligned}$$

The first inequality is obvious, the second holds by the previous lemma. The last inequality follows from the fact, that  $T^{-(r_k-1)}\tilde{E}$  is a subset of the union  $(\tilde{D} \cup \tilde{C})$ , where partition  $\mathcal{R}'$  separates each level, see Figure 6.  $\square$

#### 4. Symbolic Dynamics

In this section, we rephrase our main results in the terms of Sturmian subshifts and related sliding block codes. We use the fact that a Sturmian subshift derived from the rotation by an angle  $\alpha$  arises as coding of the rotation with respect to the partition  $\mathcal{P}$ . We also show that in other subshifts the analogous proposition need not hold.

##### 4.1. Sturmian subshifts

A *Sturmian subshift*  $(\Gamma, S)$  is the coding of the rotation with respect to the partition  $\mathcal{P}$ , i.e.,

$$\Gamma = \overline{\Phi_{\mathcal{P}}(X)} \subset \{0, 1\}^{\mathbb{N}}.$$

The topology on  $\Gamma$  is generated by the sets

$$[u]_{\ell} = \{(x_i)_{i \in \mathbb{N}} \in \Gamma \mid x_{i+\ell} = u_i \text{ for every } 0 \leq i < |u|\}, \quad u \in \{0, 1\}^*.$$

The set  $[u]_{\ell}$ , if it is nonempty, is called a *cylinder* of length  $n$  shifted by  $\ell$ . The partition of cylinders of length  $n$  shifted by  $\ell$  is defined as follows:

$$[\Sigma^n]_{\ell} = \{[u]_{\ell} \mid u \in \Sigma^n\} \setminus \{\emptyset\}.$$

The inverse mapping  $\Phi_{\mathcal{P}}^{-1}$ , applied as a set function on the subsets of  $\Gamma$ , has the following properties:

1. For every  $\ell, n \in \mathbb{N}$ , the mapping sends  $[\Sigma^n]_\ell$  bijectively onto  $T^{-\ell} \mathcal{P}^n$ , i.e.,

$$\Phi_{\mathcal{P}}^{-1}([u]_\ell) = T^{-\ell} P_u = \bigcap_{i=\ell}^{n-1+\ell} T^{-i} P_{u_i}, \quad \text{for every } u \in \Sigma^n.$$

2. The mapping preserves the relation “to be rougher than”, i.e., if  $\mathcal{R} < \mathcal{R}'$  for partitions of  $\Sigma$ , then  $\Phi_{\mathcal{P}}^{-1}(\mathcal{R}) < \Phi_{\mathcal{P}}^{-1}(\mathcal{R}')$ .

It follows immediately that the following results hold.

**Proposition 12.** *If  $n \in \mathbb{N}$  and  $\mathcal{R}$  is a nontrivial partition rougher than  $[\Sigma^n]_0$ , then there exist  $k, \ell, m \in \mathbb{N}$  such that  $\ell < n$  and*

$$\mathcal{R}^k = [\Sigma^n]_\ell.$$

**Proposition 13.** *Let  $\mathcal{R}$  be a nontrivial partition rougher than  $[\Sigma^n]_\ell$  for some  $\ell, n \in \mathbb{N}$ . Take  $\ell$  the largest and  $n$  the least possible to satisfy the assumption. If  $k \in \mathbb{N}$  such that  $r_{k-1} \leq n < r_k$ , then*

$$\mathcal{R}^{r_{k+3}+2r_k-n-2} = [\Sigma^{r_{k+3}+2r_k-3}]_\ell.$$

These results can be also reformulated in terms of sliding block codes.

#### 4.2. Sliding block codes

In this section we mainly follow the terminology from [Kû03] and [LM95]. Given a subshift  $(\Gamma, S)$  and  $m \in \mathbb{N}$ , the *language of  $\Gamma$  of length  $m$*  is the set of words defined as follows:

$$\mathcal{L}^m(\Gamma) = \{u[k, k+m) \mid u \in \Gamma, k \in \mathbb{N}\}.$$

For positive integers  $m$  and  $n$  and a mapping  $\psi$  from  $\mathcal{L}^m(\Gamma)$  to a finite set  $\Delta$ , we denote by  $\psi^{*n}$  the mapping from  $\mathcal{L}^{m+n-1}(\Gamma)$  to  $\Delta^n$  defined by the equality

$$(\psi^{*n}(u))_i = \psi(u_i u_{i+1} \dots u_{i+m-1}), \quad 0 \leq i < n, u \in \mathcal{L}^{m+n-1}(\Gamma).$$

The mapping  $\psi$  is called a *local rule of width  $m$*  and  $\psi^{*n}$  is called the *sliding block code of length  $n$  induced by  $\psi$* . In the same way, the mapping from  $\Gamma$  to  $\Delta^\mathbb{N}$  defined by the equality

$$(\psi^{*\infty}(u))_i = \psi(u_i u_{i+1} \dots u_{i+m-1}), \quad 0 \leq i < n, u \in \Gamma,$$

is the *infinite sliding block code induced by  $\psi$* .

A *homomorphism* from a subshift  $(\Gamma, S)$  to a shift  $(\Delta^{\mathbb{N}}, S)$  is any continuous mapping  $f : \Gamma \rightarrow \Delta^{\mathbb{N}}$  that commutes with shift mappings, i.e.,  $f \circ S = S \circ f$ . Since the spaces  $\Gamma$  and  $\Delta^{\mathbb{N}}$  are compact, every homomorphism  $f$  is uniformly continuous and it is therefore equal to the infinite sliding block code  $\psi^{*\infty}$  for some local rule  $\psi$ .

The main problem of this section is the relation of the injectivity of a local rule and the injectivity of the induced homomorphism. The following lemma shows that one direction follows from the definitions.

**Lemma 14.** *Let  $(\Gamma, S)$  and  $(\Delta^{\mathbb{N}}, S)$  be subshifts,  $\psi : \mathcal{L}^m(\Gamma) \rightarrow \Delta$  be a local rule. If for some  $n \in \mathbb{N}$  the sliding block code  $\psi^{*n}$  is injective, then the sliding block code  $\psi^{*(n+\ell)}$  is injective for every  $\ell \in \mathbb{N}$  and the infinite sliding block code  $\psi^{*\infty}$  is injective.*

*Proof.* Let  $\psi : \mathcal{L}^m(\Gamma) \rightarrow \Delta$  be a local rule and  $n$  be a natural number such that  $\psi^{*n}$  is injective.

Given  $\ell \in \mathbb{N}$ , suppose that words  $u$  and  $v$  from  $\mathcal{L}^{m+n+\ell-1}(\Gamma)$  have the same image under  $\psi^{*(n+\ell)}$  which we denote by  $w$ . Then for every  $i \leq \ell$ ,

$$\psi^{*n}(u[i, i + m + n - 1]) = w[i, i + n] = \psi^{*n}(v[i, i + m + n - 1]).$$

Assuming injectivity of  $\psi^{*n}$ , we get that the words  $u[i, i + m + n - 1]$  and  $v[i, i + m + n - 1]$  are the same for every  $i \leq \ell$ . It implies that  $u = v$ . This proves that  $\psi^{*(n+\ell)}$  is also injective.

The proof of injectivity of  $\psi^{*\infty}$  is analogous.  $\square$

Let us emphasize the part of the lemma which says that if a sliding block code of some finite length is injective, then the infinite sliding block code is injective too. A natural question is whether the converse holds.

If the subshift  $(\Gamma, S)$  is finite, i.e.,  $\Gamma$  is finite, then the situation is simple and the answer is affirmative. In the infinite case, an important role is played by the minimality and the dependence of the local rule on the first coordinate.

For every positive natural number  $n \geq 2$ , denote by  $g_n$  the mapping from  $\mathcal{L}^n(\Gamma)$  to  $\mathcal{L}^{n-1}(\Gamma)$  defined as the *cut-off of the last letter*, i.e.,  $g(x) = x[0, n - 1]$ . We say that a local rule  $\psi$  from  $\mathcal{L}^m(\Gamma)$  to  $\Delta$  is *minimal* if either  $m = 1$ , or there is no local rule  $\psi'$  from  $\mathcal{L}^{m-1}(\Gamma)$  to  $\Delta$  satisfying the condition:  $\psi' = \psi \circ g_m$ , i.e.,  $\psi'(u) = \psi(u[0, m - 1])$  for every  $u \in \mathcal{L}^m(\Gamma)$ . A local rule  $\psi : \mathcal{L}^m(\Gamma) \rightarrow \mathcal{L}^1(\Gamma)$  *ignores the first letter* if there exists a

mapping  $\psi' : \mathcal{L}^{m-1}(\Gamma) \rightarrow \mathcal{L}^1(\Gamma)$  such that  $\psi = \psi' \circ S$ , i.e.,  $\psi(u)$  equals  $\psi'(u[1, m))$ , for every  $u \in \mathcal{L}^m(\Gamma)$ .

A rule ignoring the first letter induces sliding block codes which ignores the first letter as well. In other words, if  $\psi = \psi' \circ S$ , then for every  $n \in \mathbb{N}$  we have  $\psi^{*n} = (\psi')^{*n} \circ S$  and  $\psi^{*\infty} = (\psi')^{*\infty} \circ S$ . Since the mapping  $S$  is not injective, the sliding block codes of all lengths and the infinite sliding block code are not injective either.

The next two lemmas show the important role of minimal local rules.

**Lemma 15.** *If  $\psi$  is a local rule of width  $m$ , then a local rule  $\psi'$  of width  $m'$  such that  $m' \leq m$  induces the same homomorphism as  $\psi$  if and only if*

$$\psi'(u[0, m')) = \psi(u), \text{ for every } u \in \mathcal{L}^m(\Gamma).$$

*In particular, given a local rule  $\psi$ , there exists just one minimal local rule  $\psi'$  of a smaller or equal width that induces the same homomorphism. The mapping  $\psi'$  is of minimal width among all mappings inducing the same homomorphism as  $\psi$ .*

The proof is straightforward.

**Lemma 16.** *Let  $\Gamma$  be infinite,  $\psi : \mathcal{L}^m(\Gamma) \rightarrow \Delta$  be a local rule of width  $m$ . If a sliding block code of finite length induced by  $\psi$  is injective, then  $\psi$  is minimal.*

*Proof.* Suppose that  $\Gamma$  is infinite,  $\psi$  is a non-minimal local rule of width  $m$  and  $n$  is a positive natural number. Since  $\psi$  is not minimal, there exists a local rule  $\psi'$  of width  $m - 1$  such that  $\psi = \psi' \circ g_m$ . It is readily seen that for every natural number  $n \in \mathbb{N}$  we have  $\psi^{*n} = (\psi')^{*n} \circ g_{m+n-1}$ . But the infiniteness of  $\Gamma$  does not allow  $g_{m+n-1}$  to be injective. Hence,  $\psi^{*n}$  is not injective either.  $\square$

In accordance with the previous discussion, we can restrict the above mentioned question as follows:

Let a minimal local rule  $\psi$  do not ignore the first letter. Does the injectivity of the sliding block code  $\psi^{*\infty}$  implies the injectivity of the sliding block code  $\psi^{*n}$  for some finite  $n \in \mathbb{N}$ ?

Example 1 shows that, in full generality, the answer to this question is negative. However, Proposition 17 gives a positive answer for Sturmian subshifts.

*Example 1.* Let  $\Gamma = \{0, 1\}^{\mathbb{N}}$  and  $\Delta = \{0, 1, 2\}$ . Let  $\psi : \mathcal{L}^2(\Gamma) \rightarrow \Delta$  be the local rule defined as follows:

$$\psi(11) = 0, \psi(10) = 0, \psi(01) = 1, \psi(00) = 2.$$

This rule is minimal and does not ignore the first letter. Let us remark that for every  $x, y \in \mathcal{L}^2(\Gamma)$ ,  $\psi(x) = \psi(y)$  implies  $x[0, 1) = y[0, 1)$ . By induction one can easily prove that for every  $n \in \mathbb{N}$ ,

$$\psi^{*n}(x) = \psi^{*n}(y) \implies x[0, n) = y[0, n).$$

Hence,  $\psi^{*\infty}(x) = \psi^{*\infty}(y)$  implies  $x = y$ . We get that the infinite sliding block code induced by  $\psi$  is injective. On the other hand, for every  $n \in \mathbb{N}$ ,  $x \in \{0, 1\}^n$ , the words  $x10$  and  $x11$  belong to  $\mathcal{L}^{n+2}(\Gamma)$  and their images under  $\psi^{*(n+1)}$  are the same. It implies that no sliding block code of finite length induced by  $\psi$  is injective.

**Proposition 17.** *Let  $(\Gamma, S)$  be a Sturmian subshift,  $\psi^{*\infty}$  be the homomorphism from  $(\Gamma, S)$  to a subshift  $(\Delta^{\mathbb{N}}, S)$  induced by a local rule  $\psi : \mathcal{L}^m(\Gamma) \rightarrow \Delta$ .*

*If the rule  $\psi$  is minimal and does not ignore the first letter, then the following conditions are equivalent:*

1.  $\psi^{*n}$  is injective for some natural number
2.  $\psi^{*\infty}$  is injective,
3.  $\psi^{*\infty}$  is not constant,
4.  $\psi$  is not constant.

*Proof.* It is readily seen that the conditions above are ordered from the strongest to the weakest, i.e. (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4). It suffices to prove that (4) implies (1). Suppose that  $\psi : \mathcal{L}^m(\Gamma) \rightarrow \Delta$  is a non-constant minimal local rule that does not ignore the first letter.

For  $a \in \Delta$ , define  $R_a = (\psi^{*\infty})^{-1}[a]$ . The set  $\mathcal{R} = \{R_a \mid a \in \Delta\} \setminus \{\emptyset\}$  forms a partition and for every  $n \in \mathbb{N}$ ,  $u \in \Delta^n$ , we get that the set  $R_u$ , defined in Section 2, satisfies the following condition,

$$\begin{aligned} R_u &= \bigcap_{k=0}^{n-1} S^{-k} ((\psi^{*\infty})^{-1}[u_k]) = \bigcap_{k=0}^{n-1} (\psi^{*\infty})^{-1}(T^{-k}[u_k]) = (\psi^{*\infty})^{-1}[u] \\ &= \bigcup \{[v] \subset \Gamma \mid v \in \mathcal{L}^{m+n-1}(\Gamma), \psi^{*n}(v) = u\}. \end{aligned}$$

In particular,  $R_u$  is a union of cylinders  $[v]$  from  $[\Sigma^m]$  for every  $u \in \Delta^*$ . Hence,  $\mathcal{R}$  is rougher than  $[\Sigma^m]$ . Let  $\ell$  be the largest and  $m'$  the least possible integer such that  $\mathcal{R}$  is rougher than  $[\Sigma^{m'}]_\ell$ .

Since  $\psi$  does not ignore the first letter, there are two words  $u, v \in \mathcal{L}^m(\Gamma)$  such that  $u[1, m) = v[1, m)$  and  $\psi(u) \neq \psi(v)$ . In particular,  $u_0 \neq v_0$ . We get that  $u$  and  $v$  are in the same set from  $[\Sigma^{m-1}]_1$ , but they are not from the same set from  $\mathcal{R}$ . Thus,  $\mathcal{R}$  is not rougher than  $[\Sigma^m]_1$ . It implies that  $\ell = 0$ . The minimality of the local rule implies that there are two words  $u, v \in \mathcal{L}^m(\Gamma)$  such that  $u[0, m-1) = v[0, m-1)$  and  $\psi(u) \neq \psi(v)$ . In particular,  $u_{m-1} \neq v_{m-1}$ . We get that  $u$  and  $v$  are in the same set from  $[\Sigma^{m-1}]_0$ , but they are not from the same set from  $\mathcal{R}$ . Thus,  $\mathcal{R}$  is not rougher than  $[\Sigma^{m-1}]_0$ . It implies that  $m' = m$ .

By Proposition 13, there exists  $n \in \mathbb{N}$  such that  $\mathcal{R}^n = [\Sigma^{m+n-1}]$ . Hence, for  $u \in \Delta^n$ ,  $R_u$  is either empty, or equal to one cylinder from  $[\Sigma^{m+n-1}]$ . It implies that there is at most one  $v \in \Sigma^{m+n-1}$  such that  $\psi^{*n}(v) = u$ . Thus,  $\psi^{*n}$  is injective.  $\square$

## 5. Open problems

First problem concerns the rotation of the unit circle and the evolution of a partition that consists of finite unions of intervals. We proved that if the endpoints of the intervals belong to the past trajectory of the point zero, then the refinements of the partition will eventually consist of connected sets, i.e., intervals. The question is if it remains to be true if we omit the assumption on the endpoints of the intervals. It is not difficult to see that it is not true in full generality. The counterexample is the partition  $\mathcal{R}$  into two sets  $[0, 1/4) \cup [1/2, 3/4)$  and  $[1/4, 1/2) \cup [3/4, 1)$ . The symmetry of the partition ensures that for every  $n \in \mathbb{N}$  and every  $x \in [0, 1/4)$ , there exist set  $M$  and  $N$  from  $\mathcal{P}^n$  such that  $M$  contains points  $x$  and  $x + 1/2$  and  $N$  contains the points  $x + 1/4$  and  $x + 3/4$ . In particular,  $M$  and  $N$  are not connected. By the same argument we can show that the counterexample is any non-trivial partition  $\mathcal{R}$  that is invariant under a rational rotation, where the invariance under a rational rotation means that there exist a natural number  $m \geq 2$  such that for every  $x \in [0, 1)$ , the number  $(x + 1/m) \bmod \mathbb{Z}$  belongs to the same set from  $\mathcal{R}$  as  $x$  does. As far as we know, it is not known whether  $\mathcal{R}^n$  eventually consists of connected sets (intervals) in the case when  $\mathcal{R}$  is not invariant under a rational rotation.

The second problem is related to the main result formulated in terms of sliding block codes. We ask for which subshifts we can answer positively the question from the previous section:

Let a minimal local rule  $\psi$  do not ignore the first letter. Does injectivity of the infinite sliding block code  $\psi^{*\infty}$  imply injectivity of the sliding block code  $\psi^{*n}$  for some finite  $n \in \mathbb{N}$ ?

In section 4.2, we discussed two poles. For subshifts of minimal subword complexity, i.e., Sturmian subshifts, the question has a positive answer. For full shifts, the subshifts of maximal subword complexity, the answer is negative. Hence, are there other classes of subshifts, likely of low subword complexity, where the answer would be also positive? A good candidate might be the class of substitution subshifts.

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